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OPTIMUM PLASTIC DESIGN OF UNBRACED FRAMES OF IRREGULAR CONFIGURATION

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Abstract—An efficient algorithm is developed for optimum plastic design of low-rise unbraced frames of general configuration. The algorithm is based on the static approach of limit analysis without generating the independent mechanisms. Three equations of equilibrium are written at each joint of the frame to form the equality constraints. To increase the efficiency of the linear programming formulation, a revised simplex method is employed in which the inverse of the basis is represented as a product of elementary matrices. The elementary matrices are actually saved as column matrices to save storage in the microcomputer. Therefore, this approach is more efficient in terms of speed and storage. Based on the algorithm presented, a microcomputer program is developed for optimum plastic design of low-rise frames of arbitrary configuration on an IBM personal computer. The objective of such an optimal design is to find plastic moment capacities of the frame elements that minimize the weight of the structure. Three examples are presented in order to show the practicality of the optimum plastic design of frames on available microcomputers.

INTRODUCTION

With increasing availability and decreasing cost of microprocessor-based computers, computer-aided design (CAD) is gaining widespread popularity. Design of members of a structure such as beams, columns and footings can be effectively performed on available popular microcomputers. Compared with mainframe computers, however, microcomputers are considerably slower and therefore may require more sophisticated and efficient algorithms for overall design of structures. One important area which needs such efficiency is optimum design of structures.

Linear programming (LP) formulation has been used since the 1960s for plastic analysis and design of structures, most of which is applicable for frames of regular configurations. The trend has been to find the basic failure mechanisms first and then combine them and perform a search for finding the collapse mechanism. Mathematically, an equilibrium equation is obtained for each independent mechanism by using the principle of virtual work and equating the external work of the applied loads with the internal work of the hinge rotations. Generation of the basic independent mechanisms for frames of complicated geometry needs extensive programming and can be very time consuming[7]. As demonstrated by Cohn *et al.*[2], a simple two-story single-bay frame has 60 possible failure mechanisms.

Most of the available literature on optimal plastic design of frames is limited to regular rectangular frames[3, 5]. In this paper, an efficient algorithm is presented for optimum plastic design of low-rise unbraced frames of general configuration. The algorithm is based on the static approach of limit analysis without generating the independent mechanisms. Based on this approach, a microcomputer BASIC program is developed for optimal plastic design of low-rise unbraced frames of general configuration on an IBM personal computer. Optimum plastic designs of three irregular frames are presented. These examples show the practicality of the optimum plastic design of frames on available microcomputers.

DEVELOPMENT OF THE EQUILIBRIUM CONSTRAINT EQUATIONS

The objective of the optimal design problem is to come up with a structure that can sustain a specified factored load and meet a certain optimality criterion. Our optimality

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criterion is to minimize the overall weight of the frame. This overall weight can be expressed in terms of the weight per unit length of the frame members which is assumed to have a linear relationship with the corresponding plastic moment capacities of the frame members. The objective function is subjected to some constraints that ensure the equilibrium of the frame and at the same time the yield condition is not violated anywhere in the frame. The problem is then solved by the linear programming technique for the plastic moment capacities of the frame members that minimize the weight function. Also, the bending moments at the critical sections are obtained.

The equilibrium constraints can be expressed by a number of equations that represent the equilibrium between the internal forces (axial and shear forces and bending moments) and the externally applied loads. These equations can be generated by considering the equilibrium of each independent mechanism of the frame. As Jones and Boaz suggested[4], the equilibrium constraint equations can be generated based on the static approach without generating the independent mechanisms.

The six end force components (end actions) for a typical frame element are shown in Fig. 1. They include two end moments M_{ij} and M_{ji}, two end shear forces V_{ij} and V_{ji} and two end axial forces P_{ij} and P_{ji}. By writing the three equilibrium equations of member ij, the end actions V_{ij}, V_{ji} and P_{ji} can be expressed in terms of the remaining three end actions:

$$V_{ij} = V_{ji} = (M_{ij}+M_{ji})/L_{ij}, \tag{1}$$

$$P_{ji} = P_{ij}. \tag{2}$$

The optimal plastic design problem is formulated by treating three member end actions M_{ij}, M_{ji} and P_{ij} for each element of the frame as variables. For any joint i, the equilibrium is ensured by summation of the forces in the global X and Y coordinates and moments about the global Z axis;

$$\sum_j -P_{ij} \cos \theta_{ij} - \sin \theta_{ij}(M_{ij}+M_{ji})/L_{ij}+F_{xi} = 0, \tag{3}$$

$$\sum_j -P_{ij} \sin \theta_{ij} + \cos \theta_{ij}(M_{ij}+M_{ji})/L_{ij}+F_{yi} = 0, \tag{4}$$

$$\sum_j -M_{ij}+M_i = 0, \tag{5}$$

where F_{xi}, F_{yi} and M_i are the external forces in the global X and Y directions and bending moment acting at joint i, respectively, L_{ij} is the length of the member ij and θ_{ij} is the angle of inclination of member ij with respect to the global X axis. Referring to Fig. 1 we can

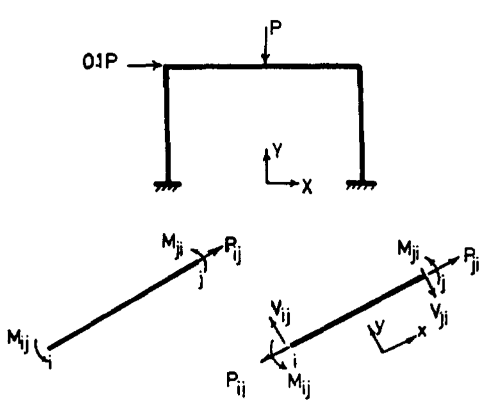

Fig. 1. End actions for a typical frame element.

write

$$\sin \theta_{ij} = (Y_j - Y_i)/L_{ij}; \qquad \cos \theta_{ij} = (X_j - X_i)/L_{ij}, \tag{6}$$

where X_i, X_j, Y_i and Y_j are the global coordinates of the joints i and j.

Now, we present a systematic and general procedure for generating the equilibrium constraint equations using matrix displacement approach. These equations can be written symbolically in the following form:

$$\mathbf{A} = \mathbf{GF}, \tag{7}$$

where $\mathbf{A}$ is the column matrix of the external loads applied at the joints, $\mathbf{F}$ is the column matrix of element internal forces (three for each element, two end moments M_{ij} and M_{ji} and axial force P_{ij} as shown in Fig. 1) and $\mathbf{G}$ is called the equilibrium coefficient matrix. Denoting the number of joints (or nodes) and elements (or members) in the frame by n and m, respectively, the size of the matrices $\mathbf{A}$, $\mathbf{G}$ and $\mathbf{F}$ are $3n \times 1$, $3n \times 3m$ and $3m \times 1$, respectively. (Note that point of application of each load is also considered a "joint" and only point loads and moments are considered in the formulation.) Matrices $\mathbf{G}$ and $\mathbf{F}$ can be written in terms of submatrices $\mathbf{G}_p$ and $\mathbf{F}_p$ for element number p in the following form:

$$\mathbf{G} = [\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_m], \tag{8}$$

$$\mathbf{F} = \{\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_m\}. \tag{9}$$

(In order to save space, a column matrix is shown in a row in braces.)

The equilibrium coefficient matrix $\mathbf{G}_p$ for element number p can be written as

$$\mathbf{G}_p = \bar{\bar{\mathbf{a}}}_p^T \mathbf{L}_p^T \bar{\mathbf{a}}_p^T. \tag{10}$$

In eqn (10), superscript T indicates the transpose of a matrix, $\mathbf{L}_p$ is the coordinate transformation matrix for element p and is written in the form

$$\mathbf{L}_p = \begin{bmatrix} \boldsymbol{\lambda} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\lambda} \end{bmatrix}, \tag{11}$$

where

$$\boldsymbol{\lambda} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{12}$$

$\bar{\mathbf{a}}_p$ is the transformation matrix relating the pth element deformation vector $\mathbf{d}_p = \{d_1, d_2, d_3\}$ corresponding to the pth element internal force vector $\mathbf{F}_p = \{f_1, f_2, f_3\}$ (see Fig. 2) and the pth element displacement vector in the local x–y axes $\bar{\mathbf{d}}_p = \{\bar{d}_{xi}, \bar{d}_{yi}, \bar{d}_{\theta i}, \bar{d}_{xj}, \bar{d}_{yj}, \bar{d}_{\theta j}\}$ as follows:

$$\mathbf{d}_p = \bar{\mathbf{a}}_p \bar{\mathbf{d}}_p, \tag{13}$$

where

$$\bar{\mathbf{a}}_p = \begin{bmatrix} 0 & 1/L & 1 & 0 & -1/L & 0 \\ 0 & 1/L & 0 & 0 & -1/L & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{14}$$

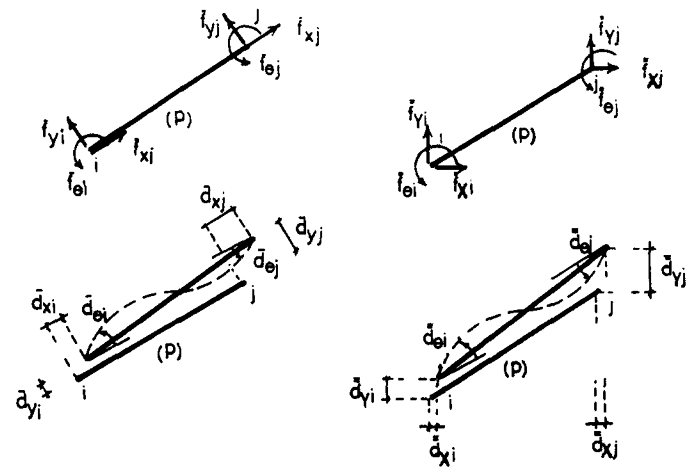

Fig. 2. Local and global end deformations and actions for an element.

Also, $\bar{\bar{\mathbf{a}}}_p$ is the transformation matrix relating the pth element displacement vector in the global X–Y axes $\bar{\bar{\mathbf{d}}}_p = \{\bar{\bar{d}}_{Xi}, \bar{\bar{d}}_{Yi}, \bar{\bar{d}}_{\theta i}, \bar{\bar{d}}_{Xj}, \bar{\bar{d}}_{Yj}, \bar{\bar{d}}_{\theta j}\}$ to the overall frame displacement vector $\mathbf{d}$ as follows:

$$\bar{\bar{\mathbf{d}}}_p = \bar{\bar{\mathbf{a}}}_p \mathbf{d}. \tag{15}$$

Note that the elements of $\bar{\bar{\mathbf{a}}}_p$ are unity when element joint displacements correspond to the overall joint displacements and zero elsewhere. The transformation matrices $\bar{\mathbf{a}}_p$, $\mathbf{L}_p$ and $\bar{\bar{\mathbf{a}}}_p$ can be easily generated and combined together subsequently to form the equilibrium matrix $\mathbf{G}$ in a computer program.

FORMULATION OF THE OPTIMUM PLASTIC DESIGN PROBLEM

Objective function

The objective in optimal plastic design of frames is to find plastic moment capacities of the members that can sustain specified load and yield the minimum frame weight. The relation between the weight per unit length of the frame member (w) and its plastic moment capacity (M_p) is in general nonlinear. Range of plastic moment capacities of the members in a low-rise frame is, however, relatively small. In this case, a linear relationship between w and M_p can be assumed and the objective function to be minimized can be defined as[6]

$$W = \sum_{i=1}^{h} M_{pi} L_i, \tag{16}$$

where M_{pi} and L_i are the plastic moment capacity and total length of members in group i, respectively. We define the design variables to be the plastic moment capacities M_{pi} ($i = 1, 2, \ldots, h$) *of* h different groups of members, and member end moments and normal forces as defined by vector $\mathbf{F}$. Each member has three unknown member end forces, two moments and one axial force, but we are interested in two end moments only.

Nonnegativity of bending moments and normal forces

The formulation of the optimal plastic design problem based on the static approach requires that equilibrium and yield conditions be satisfied. The first condition is presented by the equilibrium constraint equation (7) discussed in the previous section. The second condition simply imposes upper and lower bounds on the bending moments induced at the critical sections. These bending moments are not restricted in sign and their upper and lower bounds are the positive and negative values of the corresponding plastic moment capacities. This yield or plasticity condition can be written as

$$-\mathbf{TM}_p \leqslant \mathbf{M} \leqslant \mathbf{TM}_p, \tag{17}$$

where $\mathbf{M}$ is the vector of member end moments having $2m$ rows, $\mathbf{M}_p$ is the vector of plastic moment capacities having h rows, and $\mathbf{T}$ is a correspondence matrix with dimension $2m \times h$ that relates member end moments to the corresponding plastic moment capacities. In order to employ the standard LP formulation, the design variables should be nonnegative. To satisfy this condition, we introduce a new set of variables $\mathbf{X}$:

$$\mathbf{X} = \mathbf{M} + \mathbf{TM}_p. \tag{18}$$

Then, eqn (17) will yield $\mathbf{X} \geqslant \mathbf{0}$ and

$$\mathbf{X} - 2\mathbf{TM}_p \leqslant \mathbf{0}. \tag{19}$$

In the case of normal forces $\mathbf{P}$, two positive values are substituted for each P_i $(i = 1, 2, \ldots, m)$ as expressed by the following equation:

$$\mathbf{P} = \mathbf{P}^+ - \mathbf{P}^-. \tag{20}$$

Now, we formulate the design constraints in terms of the variables $\mathbf{X}$, $\mathbf{P}^+$ and $\mathbf{P}^-$.

Constraint equations

Having introduced the new sets of variables $\mathbf{X}$, $\mathbf{P}^+$ and $\mathbf{P}^-$, we can write the equilibrium and plasticity constraints in their final form. The relationship between the externally applied load vector $\mathbf{A}$ and the pth element internal force vector $\mathbf{F}_p$ can be written as follows:

$$\mathbf{A} = \mathbf{G}_p\mathbf{F}_p, \tag{21}$$

$$\mathbf{A} = \mathbf{G}_p\{M_{ij}, M_{ji}, P_{ij}\}. \tag{22}$$

In order to replace P_{ij} by two positive variables P_{ij}^+ and P_{ij}^-, an additional column vector is added to $\mathbf{G}_p$. Knowing that the number of columns in $\mathbf{G}_p$ is three and referring to eqn (20), the coefficients of the added column will be the negative of the coefficients of the third column. Therefore, eqn (22) can be written as

$$\mathbf{A} = \mathbf{G}_p^*\{M_{ij}, M_{ji}, P_{ij}^+, P_{ij}^-\} = \mathbf{G}_p^*\mathbf{F}_p^*, \tag{23}$$

where $\mathbf{G}_p^*$ is the pth element expanded equilibrium coefficient matrix and $\mathbf{F}_p^*$ is the expanded internal force vector. Equation (23) can also be written as follows:

$$\mathbf{A} = \mathbf{G}_p^*(\{M_{ij}, M_{ji}, 0, 0\} + \{0, 0, P_{ij}^+, P_{ij}^-\}). \tag{24}$$

Symbolically, we can write

$$\mathbf{A} = \mathbf{G}_p^*\mathbf{M}_p^* + \mathbf{G}_p^*\mathbf{P}_p^*, \tag{25}$$

where

$$\mathbf{M}_p^* = \{M_{ij}, M_{ji}, 0, 0\}; \qquad \mathbf{P}_p^* = \{0, 0, P_{ij}^+, P_{ij}^-\}.$$

Considering the equilibrium of the frame, the overall equilibrium coefficient matrix $\mathbf{G}$

is consequently expanded and its number of columns will increase from $3m$ to $4m$. Also, the bending moment vector **M** will have $2m$ additional zero elements, where two zero elements are added every two original elements to accommodate for the expanded moment vector $\mathbf{M}_p^*$. These expansions of **G** and **M** result in the matrix $\mathbf{G}^*$ and the new vector $\mathbf{M}^*$ and eqn (25) can be written for the overall frame as follows:

$$\mathbf{A} = \mathbf{G}^*\mathbf{M}^* + \mathbf{G}^*\mathbf{P}^*. \tag{26}$$

Also using $\mathbf{M}^*$ instead of **M** in eqn (18) and rearranging terms we obtain:

$$\mathbf{M}^* = \mathbf{X}^* - \mathbf{T}^*\mathbf{M}_p. \tag{27}$$

In this equation, $\mathbf{X}^*$ is the expanded **X** where 2 zero elements are added every two original elements of **X** and $\mathbf{T}^*$ is the expanded **T** where 2 zero rows are added every two rows of **T** to accommodate for the expansion of **M**. The dimensions of $\mathbf{X}^*$ and $\mathbf{T}^*$ are $4m \times 1$ and $4m \times h$, respectively. Substituting from eqn (27) into eqn (26) and rearranging terms, we obtain

$$\mathbf{A} = \mathbf{G}^*(\mathbf{X}^* + \mathbf{P}^*) + \mathbf{Z}\mathbf{M}_p, \tag{28}$$

where

$$\mathbf{Z} = \mathbf{G}^*\mathbf{T}^*.$$

The variables in the plasticity condition eqn (19) are **X** with $2m$ elements and $\mathbf{M}_p$. In order to have the same variables with the same dimension as in eqn (28) we can rewrite eqn (19) as

$$\mathbf{I}^*(\mathbf{X}^* + \mathbf{P}^*) - 2\mathbf{T}\mathbf{M}_p \leqslant 0, \tag{29}$$

where $\mathbf{I}^*$ is chosen such that $\mathbf{I}^*\mathbf{X}^* = \mathbf{X}$ and $\mathbf{I}^*\mathbf{P}^* = 0$. The matrix $\mathbf{I}^*$ is an expanded $2m \times 2m$ identity matrix where two zero columns are added every two original columns.

ALGORITHM FOR SOLUTION OF THE LP PROBLEM

The revised simplex method with the product form of the basis inverse[1] is used in this section to solve the LP problem and to obtain the required optimal solution.

Let $q = 2m + 3n$ be the total number of constraints. We introduce a set of $2m$ slack variables $\mathbf{X}_s$ to the $2m \leqslant$ inequality constraints in order to cast the LP problem in the standard form. Also, we add $3n$ artificial variables to the $3n$ equality constraints along with the $2m$ slack variables in order to form the starting basic solution. To have a basic feasible solution, all the artificial variables must have zero values. Therefore, in this work a two-phase approach is employed where at the end of phase one no artificial variables will be in the basic solution and they all will have zero values. After a feasible basic solution is found, the solution proceeds with phase two from a basic feasible solution to another one until the optimum solution is found.

Now, we can write the LP problem in its standard form as follows:

Minimize

$$W = \mathbf{L}^T\mathbf{M}_p \tag{30}$$

subjected to

$$\mathbf{I}^*(\mathbf{X}^* + \mathbf{P}^*) + 2\mathbf{T}\mathbf{M}_p + \mathbf{I}\mathbf{X}_s = 0, \tag{31}$$

$$\mathbf{Z}\mathbf{M}_p + \mathbf{G}^*(\mathbf{X}^* + \mathbf{P}^*) = \mathbf{A}, \tag{28}$$

(repeated)

$$\mathbf{M}_p, \mathbf{X}^*, \mathbf{P}^* \quad \text{and} \quad \mathbf{X}_s \geqslant 0, \tag{32}$$

where **L** is an $h \times 1$ matrix of member lengths for h groups of members and **I** is a $2m \times 2m$ identity matrix. Let us introduce the new variable W' as the negative of the objective function W. The minimization of W is the same as the maximization of the new variable W' and we can write the LP problem as follows:

Maximize W' subjected to

$$\mathbf{C}\{\mathbf{M}_p, (\mathbf{X}^* + \mathbf{P}^*), \mathbf{X}_s\} = \mathbf{B}, \tag{33}$$

$$\mathbf{L}^T\mathbf{M}_p + W' = 0, \tag{34}$$

$$\mathbf{M}_p, \mathbf{X}^*, \mathbf{P}^* \quad \text{and} \quad \mathbf{X}_s \geqslant 0. \tag{35}$$

Note that eqn (33) is a compact form for eqns (28) and (31) and the elements of the matrix **C** are the coefficients of the variables $\mathbf{M}_p$, $\mathbf{X}^*$, $\mathbf{P}^*$ and $\mathbf{X}_s$ in these two equations. Also, the vector **B** represents the right-hand side vector of the constraint equations and is equal to

$$\mathbf{B} = \{\mathbf{0}, \mathbf{A}\},$$

where **0** is a vector with $2m$ zero elements.

In each equality constraint the added artificial variable can be interpreted as a measure of the error in the original constraint equation. Adding all the equilibrium constraints to each other we obtain the equation

$$(C_{q+2,1}, C_{q+2,2}, \ldots, C_{q+2,h+4m})\begin{bmatrix} \mathbf{M}_p \\ \mathbf{X}^* + \mathbf{P}^* \end{bmatrix} + S = B_{q+2}, \tag{36}$$

where

$$C_{q+2,j} = -\sum_{i=2m+1}^{\gamma} C_{ij}; \qquad j = 1, 2, \ldots, h+4m, \tag{37}$$

and

$$B_{q+2} = -\sum_{i=2m+1}^{\gamma} A_i. \tag{38}$$

Let $C_{q+2,1}$, $C_{q+2,2}$, ... and $C_{q+2,h+4m}$ be the elements of the $q+2$nd row in the coefficient matrix **C** and B_{q+2} be the $q+2$nd element in **B**. Also let the elements of the $q+1$st row of **C** be the coefficients of the plastic moment capacities in eqn (34). The new variable S represents the negative of the absolute sum of the errors of an approximate nonnegative solution to the LP problem. S cannot be positive since all the artificial variables must be positive to fulfill the LP requirement for nonnegativity of the variables. It is also clear that when S is zero, all the artificial variables will be equal to zero. Now, the number of constraints as well as the number of variables is increased by two and the LP problem to be solved by the revised simplex method is written finally in the following form:

Maximize W' subjected to

$$\mathbf{C}\{\mathbf{M}_p, (\mathbf{X}^* + \mathbf{P}^*), \mathbf{X}_s, W', S\} = \mathbf{B}, \tag{39}$$

$$\mathbf{M}_p, \mathbf{X}^*, \mathbf{P}^* \quad \text{and} \quad \mathbf{X}_s \geqslant 0. \tag{40}$$

In matrix form, this LP problem can be written as

$$\begin{bmatrix} -2T_{(2m)\times(h)} & \mathbf{I}^*_{(2m)\times(4m)} & \mathbf{I}_{(2m)\times(2m)} \\ \mathbf{Z}_{(3n)\times(h)} & \mathbf{G}^*_{(3n)\times(4m)} & \mathbf{0}_{(3n)\times(4m)} \\ \mathbf{C}_{q+1_{(6m+h)}} & & \\ \mathbf{C}_{q+2_{(6m+h)}} & & \end{bmatrix} \begin{bmatrix} \mathbf{M}_p \\ \mathbf{X}^*+\mathbf{P}^* \\ X_s \\ W' \\ S \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{A} \end{bmatrix}, \tag{41}$$

where

$$\mathbf{C}_{q+1} = \{\mathbf{L}^T, 0, 0, \ldots, 1, 0\} \tag{42}$$

and

$$\mathbf{C}_{q+2} = \{C_{q+2,1}, C_{q+2,2}, \ldots, C_{q+2,h+4m}, 0, 0, \ldots, 0, 1\}. \tag{43}$$

Equation (41) can be written symbolically as follows:

$$\mathbf{CX} = \mathbf{B}. \tag{44}$$

The iterative solution process starts by choosing a starting basic solution. This basic solution is usually an identity matrix of dimension $(q+2)\times(q+2)$ the inverse of which is also the identity denoted by the matrix $\mathbf{U}$. The $m\times m$ submatrix in the upper left corner of $\mathbf{U}$ represents the inverse of the starting basis which corresponds to the q design variables (including any slack or artificial variables) in the basic solution. The last two rows of $\mathbf{U}$ are used to determine the entering variables where row $q+2$nd is used in phase I and row $q+1$st is used in phase II.

Going from one basic solution to another one, an interchanging process between the variables is performed. To improve the solution, one of the basic variables is replaced by a nonbasic one which becomes automatically basic. The leaving and the entering variables are denoted by X_L, and X_K, respectively. An updating process is followed for the inverse of the basis matrix $\mathbf{U}$ as well as the right-hand side vector $\mathbf{B}$. This is done by conducting an elimination process on the rows of $\mathbf{U}$ knowing the pivoting element c_{LK} in the coefficient matrix $\mathbf{C}$. The updated coefficients u'_{ij} of $\mathbf{U}$ can be expressed in terms of the current coefficients u_{ij} as

$$u'_{ij} = u_{ij} - (u_{Lj}/a_{Lk})a_{ik} \qquad \text{for} \qquad i \neq L, \tag{45}$$

$$u'_{ij} = u_{Lj}/a_{Lk} \qquad \text{for} \qquad i = L, \tag{46}$$

and for the whole matrix $\mathbf{U}$, these two equations can be written as

$$\mathbf{U}' = \mathbf{EU}, \tag{47}$$

where $\mathbf{U}'$ is the updated inverse of the basis and $\mathbf{E}$ is an elementary matrix defined to be a square matrix that differs from the identity in only one column denoted by the vector $\mathbf{Y}$. From the elimination formulas (45) and (46), the elements of the $\mathbf{Y}$ vector can be written as

$$y_i = -x_{ik}/a_{Lk} \qquad \text{for} \qquad i \neq L, \tag{48}$$

$$y_L = 1/a_{Lk} \qquad \text{for} \qquad i = L. \tag{49}$$

Then, E can be written as follows:

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & . & . & y_1 & . & 0 \\ 0 & 1 & . & . & y_2 & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & y_L & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & y_m & . & 1 \end{bmatrix}.$$

In order to save computer storage, the matrix **E** can be stored as the column vector **Y** knowing its position L in **E**. Therefore, **E** can be represented in the following form:

$$\mathbf{E} = \{L, y_1, y_2, \ldots, y_L, \ldots, y_{q+2}\}. \qquad (50)$$

Denoting the inverse of the first initial basis by $\mathbf{U}_1$, the basis inverse in the second iteration is given by

$$\mathbf{U}_2 = \mathbf{E}_1\mathbf{U}_1 = \mathbf{E}_1, \qquad (51)$$

where $\mathbf{U}_1$ is an identity matrix representing the starting basic solution. Therefore, after t iterations the inverse of the basis can be written as

$$\mathbf{U}_t = \mathbf{E}_{t-1}\mathbf{E}_{t-2}\ldots\mathbf{E}_2\mathbf{E}_1 = \mathbf{E}_{t-1}\mathbf{U}_{t-1} \qquad (52)$$

or

$$\mathbf{U}' = \mathbf{E}_{t-1}\mathbf{U}. \qquad (53)$$

Taking advantage of the fact that **E** is an elementary matrix, the product in eqn (53) can be written as

$$u'_{ij} = u_{ij} + y_i u_{Lj} \quad \text{for} \quad i \neq L, \qquad (54)$$

$$u'_{Lj} = y_L u_{Lj} \quad \text{for} \quad i = L. \qquad (55)$$

Also, the updated right-hand side vector can be written as

$$\mathbf{B}' = \mathbf{U}'\mathbf{B} = \mathbf{E}_{t-1}\mathbf{U}_{t-1}\mathbf{B}_{t-1} = \mathbf{E}_{t-1}\mathbf{B}. \qquad (56)$$

Based on the algorithm presented in this paper, a microcomputer program is developed for optimum plastic design of low-rise unbraced frames of general configuration.

MICROCOMPUTER PROGRAM

An interactive microcomputer BASIC program is developed and compiled on an IBM Personal Computer. The program is composed of a main routine and three subroutines and consists of 650 lines. The data to be provided by the user are the geometric properties of the frame as well as the loading condition and are read through the main part of the program and the first subroutine (subroutine DATA). The second subroutine (EQUIL) generates the constraint coefficients matrix C. The optimization problem is solved by the third subroutine (LINPRO) which performs the steps of the revised simplex method with the product form of the basis inverse. The program outputs are the plastic moment capacities of the different groups of members, the value of the objective function and member end moments for each frame member. The program can handle hinged, roller and fixed supports.

EXAMPLES

Three examples are presented in this section. The first example, shown in Fig. 3, is a two-story, two bay unsymmetrical frame. The loading on the members as well as member grouping is shown in Fig. 3. The optimum plastic moment capacities found for this example are

$$M_{p1} = 38.1818\,\text{K}\cdot\text{ft}, \qquad M_{p2} = 76.3636\,\text{K}\cdot\text{ft}, \qquad M_{p3} = 97.2727\,\text{K}\cdot\text{ft}.$$

The bending moment diagram for the optimum frame is presented in Fig. 4.

The second example is also a two-story frame with inclined members shown in Fig. 5. The optimum moment capacities for the different groups of members were found to be

$$M_{p1} = 4.1874\,\text{PL}, \qquad M_{p2} = 6.594\,\text{PL}, \qquad M_{p3} = 2.4\,\text{PL}, \qquad M_{p4} = 2.5\,\text{PL}.$$

The bending moment diagram for the optimum frame is presented in Fig. 6.

The third example is a four-story unsymmetrical frame shown in Fig. 7. This relatively large LP problem has 92 constraints (44 inequality constraints and 48 equality or equilibrium

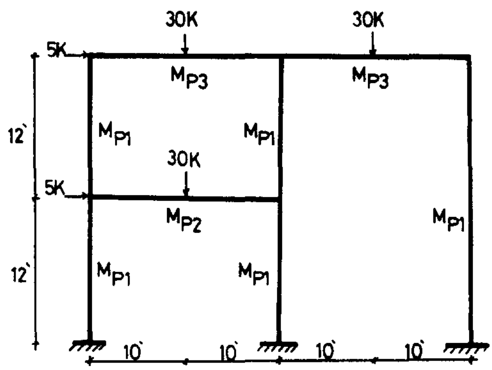

Fig. 3. Frame of example one.

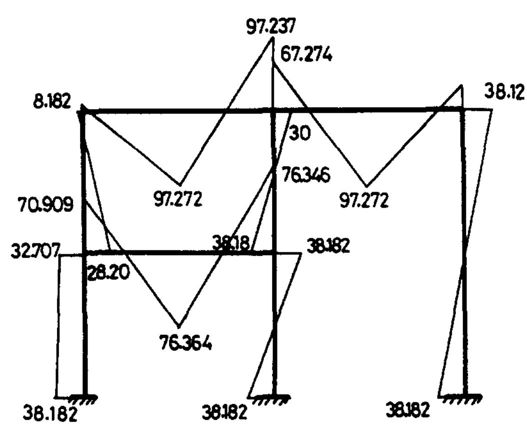

Fig. 4. Bending moment diagram for example one.

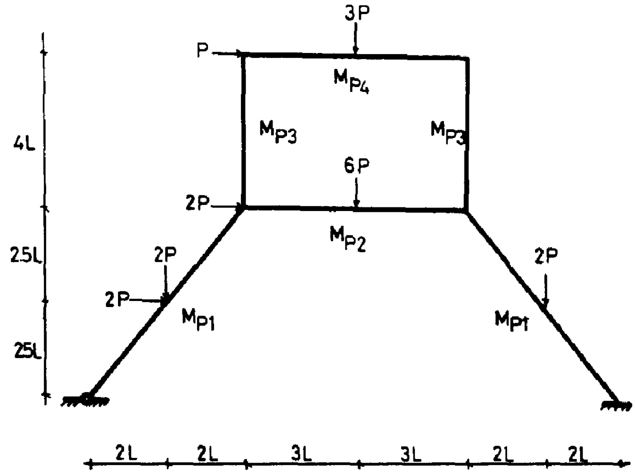

Fig. 5. Frame of example two.

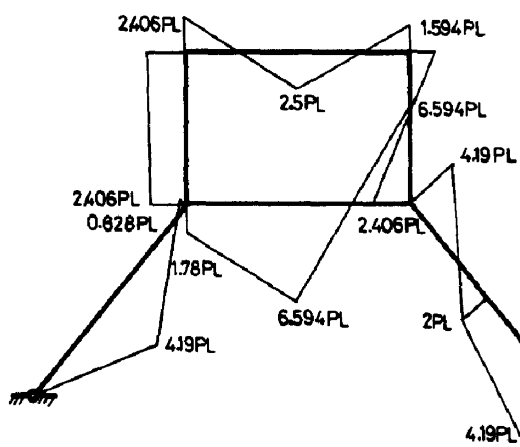

Fig. 6. Bending moment diagram for example two.

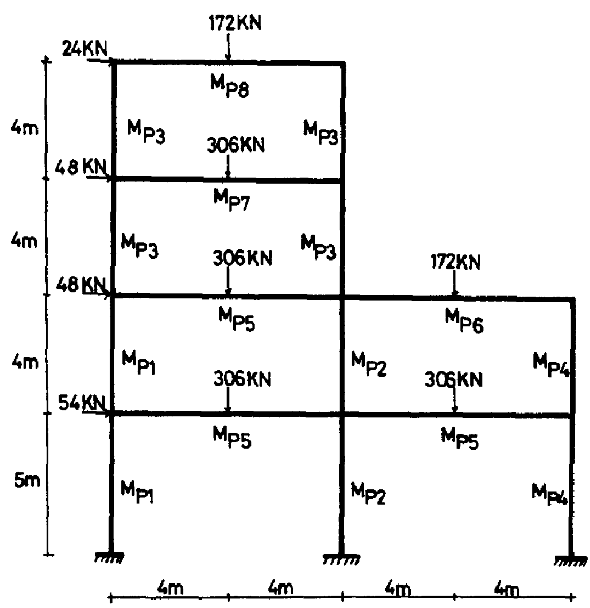

Fig. 7. Frame of example three.

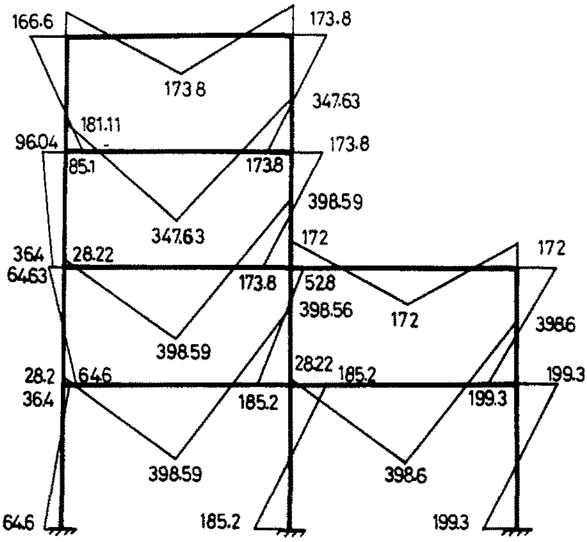

Fig. 8. Bending moment diagram for example three.

constraints). The optimum plastic moment capacities found for this example are

$$M_{p1} = 64.629\,\text{KN}\cdot\text{m}, \quad M_{p2} = 185.185\,\text{KN}\cdot\text{m}, \quad M_{p3} = 173.815\,\text{KN}\cdot\text{m},$$
$$M_{p4} = 199.296\,\text{KN}\cdot\text{m}, \quad M_{p5} = 398.593\,\text{KN}\cdot\text{m}, \quad M_{p6} = 172.000\,\text{KN}\cdot\text{m},$$
$$M_{p7} = 347.630\,\text{KN}\cdot\text{m}, \quad M_{p8} = 173.815\,\text{KN}\cdot\text{m}.$$

The bending moment diagram for this example is presented in Fig. 8.

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